U.G. 5th Semester Examination - 2020
MATHEMATICS
[HONOURS]
Course Code : MATH-H-CC-T-12

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

The notations and symbols have their usual meanings.

1. Answer any ten questions. $2 \times 10 = 20$

With proper justification state whether the following statement is true or false. No credit will be given if only true/false is written without any proper justification.

(a) A cyclic group of order 6 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$.

(b) Let $\text{Aut}(\mathbb{Z}_7)$ denote the group of automorphisms of $(\mathbb{Z}_7,+)$. Then $\text{Aut}(\mathbb{Z}_7)$ is a non-commutative group.

(c) Let $\text{Aut}(\mathbb{Z})$ denote the group of automorphisms of the group $(\mathbb{Z}, +)$. Then $\text{Aut}(\mathbb{Z})$ is isomorphic to the group $(\mathbb{Z}, +)$.

(d) $\mathbb{Z}_2 \times \mathbb{Z}_6$ and $\mathbb{Z}_2 \times S_3$ are isomorphic groups.

(e) The additive group $(\mathbb{Z}, +)$ can not be expressed as an internal direct product of two non trivial subgroups of $(\mathbb{Z}, +)$.

(f) Let G be a finite group that has only two conjugate classes. Then G is isomorphic to $(\mathbb{Z}_3, +)$.

(g) The order of a Sylow 2-subgroup of the group $(\mathbb{Z}_{12},+)$ is 2.

(h) Every group of order 15 is non cyclic.

(i) Every group of order 4 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(j) The symmetric group S_3 does not contain any Sylow 3-subgroup.

(k) The number of elements of U_{11} (the group of units modulo 11) is 10.

(l) Up-to isomorphism, there exist exactly 3 commutative groups of order 8.

(m) The alternating group A_3 is a simple group.

(n) Let p be a prime integer and G be a finite group such that p divides the order of G. Then any two p-Sylow subgroups of G are isomorphic.

(o) Every non commutative group of order 21 contains a subgroup of order 3.

2. Answer any four questions.

   (a) Define a left action of the symmetric group S_3 on the set S=$\{1, 2, 3\}$. Find all distinct orbits of S_3 with respect to the above defined group action. $2 + 3 = 5$

   (b) Let G be a group of order 14=2×7. Show that G has a normal subgroup of order 7. 5

   (c) Let G be a group of order p^2, where p is a prime integer. Prove that G is a commutative group. 5

   (d) Define commutator subgroup G' of a group G. Show that the commutator subgroup G' of the group G is a normal subgroup of G. $2 + 3 = 5$

   (e) Up-to isomorphism, find all abelian groups of order 36. 5

   (f) Let G be a cyclic group of order mn, where m and n are relatively prime positive integers i.e. $gcd(m, n) = 1$. Prove that $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$. 5

3. Answer any two questions.

   (a) Let G be a group and S be a nonempty subset of G. Let G acts on S. Define a relation ρ on S by for all $a, b \in S$, $a\rho b$ if and only if $a = gb$ for some $g \in G$.

      i. Prove that ρ is an equivalence relation on S.

      ii. For a∈S, define $G_a = \{g \in G : ga = a\}$. Prove that G_a is a subgroup of G.

      $5 + 5 = 10$

   (b) i. Find the class equation of the symmetric group S_3.

      ii. Show that a group of order 8 can not be a simple group.

      $5 + 5 = 10$

   (c) i. Let p and q be two primes. Prove that no group of order pq is simple.

      ii. Prove that up-to isomorphism, there exists only one group of order 77.

      $5 + 5 = 10$

   (d) i. Prove that $Aut(\mathbb{Z}_n) \simeq U_n$, where $Aut(\mathbb{Z}_n)$ denote the group of automorphisms of $\mathbb{Z}_n$ and U_n denotes the group of units modulo n.

      ii. Let G be a group and $Inn(G)$ denote the set of all inner automorphisms of G. Prove that $Inn(G)$ is a normal subgroup of $Aut(G)$.

      $5 + 5 = 10$